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The Virtual Mass of a Growing and Collapsing Bubble

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Analytical solutions were obtained for the virtual mass coefficient (C_V) *of a growing and collapsing bubble controlled by the following causes: thermal diffusion and mass diffusion. Expressions were derived for* C_V *of bubble growth and collapse controlled by thermal diffusion, at high and low Jakob numbers. The solutions were validated by their asymptotic approach to the limit of the constant volume bubble as well as to the limit of the single growing bubble in a system of two growing bubbles. Some avenues for future research were pointed out.* © 2006 American Institute of Chemical Engineers *AIChE J,* 52: 2013–2019, 2006
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Introduction

Studies of bubble growth rates are important in the fields of boiling, flash evaporation, and cavitation. Vapor bubbles are formed in boiling, flashing, and cavitation. In all cases, bubbles experience acceleration during growth or deceleration during collapse due to their increase or decrease in size.

Basically, there are three different mechanisms that control the rate of growth of a bubble. These mechanisms are referred to as inertia-controlled, thermal diffusion-controlled, and mass diffusion-controlled growth. Theoretical studies of the thermal diffusion-controlled growth of a vapor bubble in a uniformly superheated liquid have been reported earlier by Plesset and Zwick,[1] Forster and Zuber,[2] Scriven,[3] and Bankoff[4]. All the results agree that the bubble grows as a function proportional to the square root of time. Theoretical correlation for a bubble growth in a uniformly superheated liquid, applicable for the entire range of the growth curve (including inertia-controlled and diffusion-controlled growth, respectively) was developed by Mikic et al.[5]

The prediction of bubble rise, drag, heat, and mass transfer rates becomes much more complex when the motion is unsteady, in which case it is necessary to introduce the impulse, virtual, or "added" mass explicitly into the equation of motion. The momentum of this mass results from the acceleration or deceleration imparted to the liquid around the growing or collapsing bubble in the flow field.

Wallis[6] stated that if a bubble changes its shape, its virtual mass changes. Accordingly, Kendoush[7] derived an expression for the virtual mass of the spherical-cap bubble. Askovic[8] derived a potential flow equation for the simultaneously accelerating and growing bubble. He considered the virtual mass effects in the equation of motion by taking the classical value of half for the virtual mass coefficient.

The significant contribution to the understanding of thermohydrodynamics of the growing and collapsing bubble derives from two careful analytical and numerical studies by Professor Jacques Magnaudet and his colleagues at the Institute de Mécanique des fluids de Toulouse.[9,10] These articles considered all possible forces encountered during growing and collapsing bubbles, namely, viscous, virtual mass, buoyancy, and history.

Ohl, Tijink, and Prosperettti[11] performed elegant experiments to explore the effects of the impulse on an expanding bubble. They found that the value of ½ for the C_V gave a good fit to their experimental data. This article will be discussed later. The collapse of a vapor bubble is commonly encountered when the bubble rises in subcooled liquid or in a high pressure region. This phenomenon is familiar in many applications, such as surface boiling, direct heating of liquid with vapor, carryunder of bubbles in circulation loops, vapor suppression, and so on.

Prosperetti[12] appreciated the existence of the added mass

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coefficient for the radius-changing bubble in the general structure of the momentum equation of gas-liquid flow, but without specifying the type of the coefficient. Kim[13] derived expressions for C_V of a bubble moving through a pipe by using potential flow theory. The derived C_V were functions of the bubble diameter.

A recent publication in this field [14] stated, "Surveying previous work as outlined above reveals that a full understanding of the mechanisms that govern bubble growth has not been reached."

The analysis of the momentum due to the added mass of a growing or collapsing bubble is rather difficult by the fact that knowledge of these concepts is limited. It is the purpose of this article to throw additional light on this problem.

Theoretical Analysis of the Virtual Mass

The velocity potential of a growing or collapsing spherical bubble of radius a(t), where t represents time, when its center moves with an instantaneous velocity U in a straight line is given as follows[15]:

$$\phi = \pm \frac{a^2 \dot{a}}{r} + \frac{1}{2} U \frac{a^3}{r^2} \cos \theta \tag{1}$$

where $\dot{a}$ is written for da/dt, which is the time rate of increase of bubble radius and θ is the angle between the radius vector r and the direction of motion. The positive sign in Eq. 1 is for bubble growth and the negative sign is for bubble collapse. Batchelor[16] stated that the radial acceleration of the growing bubble produces an irrotational velocity distribution.

The total kinetic energy of the fluid, of density ρ_f, surrounding the growing bubble is given by the following[17]:

$$KE = -\frac{1}{2} \rho_f \int_0^{\pi} \left(\phi \frac{\partial \phi}{\partial r} \right)_{r=a} dS \tag{2}$$

where S denotes the surface area of the spherical bubble. The surface dS of an elementary region about the "$\theta = 0$" line through the bubble is given by the product of its perimeter $2\pi a \sin \theta$ and width $ad\theta$. Substituting Eq. 1 and its derivative into Eq. 2 and performing the integration yields

$$KE = \pi \rho a^3 \left[2(\dot{a})^2 + \frac{1}{3} U^2 \right] \tag{3}$$

As $\dot{a} \rightarrow 0$, we recover the expression $1/3\pi \rho_f a^3 U^2$ of the constant volume single spherical bubble (see Milne-Thomson,[18]).

If we put $KE = (1/2)MU^2$, then the virtual mass of the growing bubble becomes the following:

$$M = 2\pi \rho_f a^3 \left(\frac{2(\dot{a})^2}{U^2} + \frac{1}{3} \right) \tag{4}$$

which represents the mass of the fluid moving with a velocity U that has the same kinetic energy as the unbounded fluid around the growing bubble. The virtual mass coefficient could be obtained straightforwardly as follows: Manipulating the right side of Eq. 4 yields:

$$M = 2\pi \rho_f a^3 \left(\frac{2(\dot{a})^2}{U^2} + \frac{1}{3} \right) \left(\frac{2}{3} \right) \left(\frac{3}{2} \right) = \frac{4}{3} \pi \rho_f a^3 \left(\frac{3(\dot{a})^2}{U^2} + \frac{1}{2} \right) \tag{5}$$

This procedure was adopted by Lamb[17] and Milne-Thomson[18]; hence, we get:

$$M = mC_V \tag{6}$$

where m is the mass of the fluid displaced by the growing spherical bubble; therefore,

$$C_V = \frac{3(\dot{a})^2}{U^2} + \frac{1}{2} \tag{7}$$

This is a new equation that has not been reported before. As $\dot{a} \rightarrow 0$, C_V given by this equation reduces to ½, the classical value for a spherical bubble with constant size. This is a general equation that can be applied to bubble growth controlled by either thermal diffusion or mass diffusion. For thermal diffusion-controlled growth and at high Jakob numbers (that is, $Ja > 16$), Riznic et al.[19] recommended the following equation for the bubble growth under the following assumptions:

- moving boundary during phase change.
- neglecting effects of inertia and of surface tension.
- time dependent interface temperature.

$$a = \pi Ja \left(\frac{t\kappa}{3} \right)^{1/2} \tag{8}$$

Jakob number is defined as ($Ja = \rho_f c \Delta T / \rho_g h_{fg}$) where c is the specific heat, ΔT is the superheat, ρ_g is the saturated vapor density, h_{fg} is the latent heat of vaporization, and κ is the thermal diffusivity. The differentiation of Eq. 8 gives

$$\frac{da}{dt} = \frac{\pi}{2} Ja \left(\frac{\kappa}{3t} \right)^{1/2} \tag{9}$$

Substituting Eq. 9 into Eq. 7 yields

$$C_V = \frac{(\pi Ja)^2}{4Pe(t)\tau} + \frac{1}{2} \tag{10}$$

as the virtual mass coefficient for a growing bubble at high Jakob number. Here the Peclet number is ($Pe(t) = 2a(t)U/\kappa$), which characterizes the importance of convective heat transfer due to bubble translation relative to conduction and $\tau = tU/2a(t)$ is the dimensionless time.

For low Ja (that is, $Ja < 1$), Riznic et al.[19] proposed the following equation:

$$a = (2\kappa Jat)^{1/2} \tag{11}$$

Differentiating this equation and substituting it into Eq. 7 gives

$$C_V = \frac{3Ja}{2Pe(t)\tau} + \frac{1}{2} \tag{12}$$

as the virtual mass coefficient for a growing bubble at low *Ja*.

The collapse is the final stage of the bubble lifetime. Bubbles originate by a process called nucleation, then they grow in size, and finally they collapse when the local hydrodynamic pressure forces become high in comparison to the pressure inside the bubble.

Consider the rise of a collapsing bubble through liquid. The equation of the heat-transfer controlled collapse is given by Florschuetz and Chao[20] as follows:

$$a = a_o - 2Ja\left(\frac{t\kappa}{\pi}\right)^{1/2} \tag{13}$$

Here a_0 is the equilibrium radius of the bubble at uniform superheating. This equation was derived from solving Rayleigh's equation, the energy equation, and the continuity equation of vapor inside the bubble. The solution was approximated using the following assumptions:

- heat transfer dominates the collapse
- inertia effects were deleted from Rayleigh's equation
- spherical symmetry collapse in an incompressible liquid.

The differentiation of Eq. 13 gives the following:

$$\frac{da}{dt} = -Ja\left(\frac{\kappa}{\pi t}\right)^{1/2} \tag{14}$$

Substituting Eq. 14 into Eq. 7 yields

$$C_V = \frac{3Ja^2}{\pi Pe(t)\tau} + \frac{1}{2} \tag{15}$$

as a virtual mass coefficient of a collapsing bubble.

Note that the negative sign in Eq. 1 gives the same results for the kinetic energy and C_V as those of the bubble growth. Carefully designed experiments need to be performed in the future for the purpose of measuring the virtual mass coefficients of a growing or collapsing bubble given by Eqs. 10, 12, and 15. These experiments should be akin to those of Stelson and Mavis[21] and Iverson and Balent[22].

The New Equation of Motion of the Growing Bubble

Assume a spherical bubble growing and accelerating rectilinearly in an infinite, quiescent, and superheated liquid. The unsteady motion of the growing bubble is subjected to the following forces:

- Buoyancy force F_B due to density difference.
- Drag force F_D on the surface of the bubble.
- The impulse due to the virtual mass of the bubble.

The equation of motion can thus be written as follows:

$$\frac{d}{dt}[(m_b + M)U] = F_B - F_D \tag{16}$$

where

m_b = mass of the bubble = $(4/3)\pi a^3 \rho_g$

$$F_D = 4\pi a \mu_f\left[U + \frac{2}{U}\left(\frac{da}{dt}\right)^2\right]$$

The above equation is derived in the Appendix and will be discussed later.

$$F_B = (4/3)\pi a^3 g(\rho_f - \rho_g)$$

Substituting Eq. 6 into Eq. 16, performing the differentiation, and rearranging yields the following equation:

$$4\pi a^2\rho_g U\frac{da}{dt} + \frac{4}{3}\pi a^3\rho_g\frac{dU}{dt} + \frac{12\pi a^2\rho_f}{U}\left(\frac{da}{dt}\right)^3 + \frac{8\pi a^3\rho_f}{U}\left(\frac{da}{dt}\right)\left(\frac{d^2a}{dt^2}\right) - \frac{4\pi a^3\rho_f}{U^2}\left(\frac{da}{dt}\right)^2\frac{dU}{dt} + 2\pi a^2\rho_f U\frac{da}{dt} + \frac{2}{3}\pi a^3\rho_f\frac{dU}{dt} = \frac{4}{3}\pi a^3 g(\rho_f - \rho_g) - 4\pi a\mu_f\left(U + \frac{2}{U}\left(\frac{da}{dt}\right)^2\right) \tag{17}$$

This is a general equation of motion that could be applied to bubble growth by thermal and mass diffusion. We shall consider the case of the bubble growth by thermal diffusion at low *Ja*.

The twice differentiation of Eq. 11 gives

$$\frac{d^2a}{dt^2} = -\frac{1}{2}\left(\frac{\kappa Ja}{2}\right)^{1/2} t^{-3/2} \tag{18}$$

The drag force on a growing bubble at low *Ja* is given by Kendoush[23] as follows:

$$F_D = 4\pi\mu_f(2\kappa Jat)^{1/2}\left[U + \frac{\kappa Ja}{tU}\right] \tag{19}$$

Substituting Eq. 11, its first derivative, and Eqs. 18 and 19 into Eq. 17, dividing throughout by $2\pi a^2(\kappa Jat)^{1/2}$, and rearranging yields the following equation:

$$\frac{dU}{dt}\left[\frac{2\sqrt{2}}{3}\rho_g + \frac{\sqrt{2}}{3}\rho_f - \frac{\sqrt{2}\,\rho_f\kappa Ja}{tU^2}\right] + \frac{U}{t}\left[\sqrt{2}\,\rho_g + \frac{\rho_f}{\sqrt{2}} + \frac{\sqrt{2}\,\mu_f}{\kappa Ja}\right] + \frac{1}{t^2U}[0.707\rho_f\kappa Ja + \sqrt{2}\,\mu_f] = \frac{2\sqrt{2}}{3}g(\rho_f - \rho_g) \tag{20}$$

This is a nonlinear ordinary differential equation of motion of the growing bubble at low *Ja*. In a similar method to that

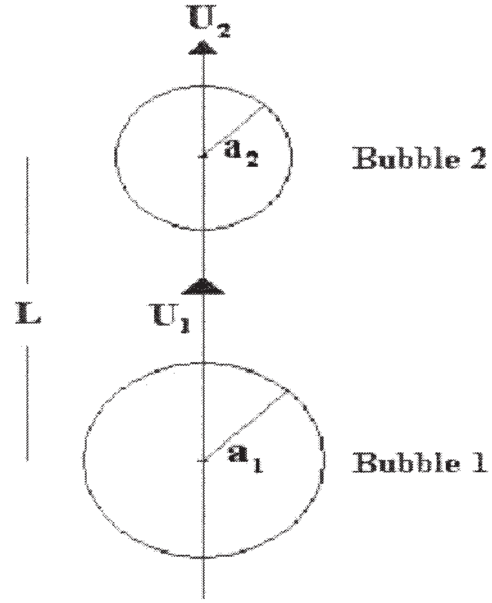

Figure 1. Two growing bubbles moving in line.

used before, the equation of motion of the growing bubble at high Ja becomes the following:

$$\frac{dU}{dt}\left[\frac{2\rho_g}{3}+\frac{\rho_f}{3}-\frac{\pi^2 Ja^2\kappa\rho_f}{6tU^2}\right]+\frac{U}{t}\left[\rho_g+\frac{\rho_f}{2}+\frac{6\mu_f}{\pi\kappa Ja^2}\right]+\frac{1}{t^2U}\left[0.0833\pi^2Ja^2\kappa\rho_f+\pi\mu_f\right]=\frac{2}{3}g(\rho_f-\rho_g) \quad (21)$$

Validation and Discussion

The analyses below are for validating the present results and for deriving a new expression for the virtual mass of a growing bubble within a system of two bubbles rising along their line of centers. Oguz and Prosperetti[24] derived the following velocity potential for two growing bubbles moving in the direction of their line of centers as shown in Figure 1:

$$\phi_1 = A+\frac{B}{r}+Cr\cos\theta+\frac{E\cos\theta}{r^2} \quad (22)$$

where

$$A=-\frac{\dot a_2 a_2^2}{L}+\frac{U_2 a_2^3}{2L^2},\ B=-a_1^2\dot a_1,\ C=-\frac{a_2^2 a\dot a_2}{L^2}+\frac{U_2a_2^3}{L^3},$$

$$E=-\frac{1}{2}U_1a_1^3$$

The kinetic energy of the fluid surrounding bubble 1 is given by Eq. 2 and, when we differentiated Eq. 22 and substituted it into Eq. 2, we obtained the following expression for the kinetic energy:

$$KE_1=\pi\rho a_1^3\left[\frac{2\dot a_1\dot a_2a_2^2}{a_1L}-\frac{U_2a_2^3\dot a_1}{a_1L^2}+2\dot a_1^2+\frac{1}{3}U_1^2-\frac{2}{3}F+\frac{U_1a_2^2\dot a_2}{3L^2}-\frac{U_1U_2a_2^3}{3L^3}\right] \quad (23)$$

where

$$F=\left(\frac{\dot a_2^2a_2^4}{L^4}-\frac{2\dot a_2U_2a_2^5}{L^5}+\frac{U_2^2a_2^6}{L^6}\right)$$

Applying the previous method of solution to get the virtual mass of bubble 1 in the presence of bubble 2, it follows that:

$$M=2\pi\rho a_1^3\left[\frac{2\dot a_1\dot a_2a_2^2}{a_1LU_1^2}-\frac{U_2a_2^3\dot a_1}{a_1L^2U_1^2}+\frac{2\dot a_1^2}{U_1^2}+\frac{1}{3}-\frac{2}{3}F+\frac{a_2^2\dot a_2}{3L^2U_1}-\frac{U_2a_2^3}{3U_1L^3}\right] \quad (24)$$

Accordingly, the virtual mass coefficient of bubble 1 becomes the following:

$$C_V=\frac{3}{2}\left[\frac{2\dot a_1\dot a_2a_2^2}{a_1LU_1^2}-\frac{U_2a_2^3\dot a_1}{a_1L^2U_1^2}+\frac{2\dot a_1^2}{U_1^2}+\frac{1}{3}-\frac{2}{3}F+\frac{a_2^2\dot a_2}{3L^2U_1}-\frac{U_2a_2^3}{3U_1L^3}\right] \quad (25)$$

As $L\to\infty$, we recover the derived Eq. 7 of the solitary growing bubble.

For validating the present results, a comparison was made with the experimental data of Darby,[25] who measured bubble growth and trajectories relative to a nucleation site using boiling water and Freon-113 as working fluids. Uniform superheat was maintained by infrared heating. Darby's experimental conditions are well suited for comparison with the low Ja results of the present theory. The comparison is shown in Kendoush,[23] where a good agreement was obtained between the present solution of the equation of motion (Eq. 20) and the experimental data of Darby.[25] In Figures 7 and 8 of Kendoush,[23] the bubble trajectory was plotted against time at $Ja = 4.8$ and 6.8, respectively.

Unrealistic solutions were obtained when we tried to use the following parameters in the numerical solution of the equation of motion (Eq. 16):

(i) Steady drag force (that is, $F_D = 12\pi a\mu_f U$ which corresponds to $C_D = 48/\mathrm{Re}$).

(ii) Virtual mass coefficient of ½.

The classical solution ($C_V = 1/2$) must be applicable with a reasonable accuracy to growing or collapsing bubbles, as far as $|\dot a| \ll U$.

Takahashi et al.[26] performed experiments for the measurements of the virtual mass coefficient and found it to vary inversely with the Re. This is in agreement with the present theoretical results, but comparison was not possible as they measured the virtual mass coefficient of an oscillating spherical particle in fluid. Note that Pe in Eqs. 10, 12, and 15 is the product of Re times the Prandtl number.

Ohl, Tijink, and Prosperettti[11] performed experiments where they injected an air bubble into a pressurized water column and watched the bubble growth during depressurization. They varied C_V from 0.4 to 0.6 in a trial to fit the experimental data. The following is a comparison between the present theoretical results and the experimental data of Ohl et al.[11] that were considered as a mass diffusion–controlled bubble growth. The depressurization of their experimental system produced elastic waves propagating along the test column. This induced a large acceleration in the liquid (about 1.5 ms^{-2}) and caused a slight reduction in the column diameter. The liquid velocity U_L was included into the equation of motion as follows:

$$\frac{d}{dt}\left[(m_b + M)(U - U_L)\right] = F_B - F_D + \frac{4}{3}\pi a^3 \rho_f \frac{dU_L}{dt} \quad (26)$$

Data were presented in their Figure 12 that enabled the estimation of the liquid velocity. The data were fitted to the following polynomial function:

$$U_L = 3.11 \times 10^9 t^9 - 2.175 \times 10^9 t^8 + 6.393 \times 10^8 t^7 - 1.024 \times 10^8 t^6 + 0.966 \times 10^5 t^5 - 5.427 \times 10^5 t^4 - 1.743 \times 10^4 t^3 - 288 t^2 + 1.56 \times 10^{-5} \quad (27)$$

A prefactor of 0.8 was multiplied by this equation taking into account the diameter of the column and the position of the column wall displacement. U_L in Eq. 27 is in units of m/s and time is in seconds.

The mass of the bubble m_b was neglected during the integration of Eq. 26 as was done by Ohl et al.[11] Considering a relative velocity $U - U_L$ into Eq. 5 and substituting F_D of Eq. 16 into Eq. 26 yields the following:

$$\frac{d}{dt}\left[\frac{2}{3}a^2\left(\frac{3(\dot{a})^2}{(U - U_L)^2} + \frac{1}{2}\right)(U - U_L)\right] = \frac{2}{3}a^2 g - 2\frac{\mu_f}{\rho_f}\left[U - U_L + \frac{2}{U - U_L}\left(\frac{da}{dt}\right)^2\right] + \frac{2}{3}a^2\frac{dU_L}{dt} \quad (28)$$

This equation was further simplified to give the following:

$$\frac{dU}{dt}\left[\frac{a}{6} - a\left(\frac{da}{dt}\right)^2 (U - U_L)^{-2}\right] = \frac{a}{2}\frac{dU_L}{dt} + \frac{a}{3}g - \frac{\mu_f}{a\rho_f}\left[U - U_L + \frac{2}{U - U_L}\left(\frac{da}{dt}\right)^2\right] - \left(\frac{da}{dt}\right)^3 (U - U_L)^{-1} - \frac{1}{3}\left(\frac{da}{dt}\right)(U - U_L) - 2a\left(\frac{da}{dt}\right)\left(\frac{d^2a}{dt^2}\right)(U - U_L)^{-1} - \frac{a}{(U - U_L)^2}\left(\frac{da}{dt}\right)^2\frac{dU_L}{dt} \quad (29)$$

The radius of the bubble was obtained from the experimental data of Figure 4 of Ohl et al. These data were fitted to the following function:

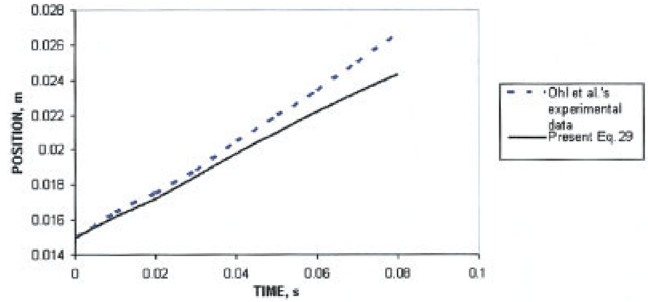

Figure 2. A comparison of air bubble trajectory of Ohl et al.'s experiments with the present solution.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

$$a = 0.000514 \exp(4.5669t) \quad (30)$$

The dimensions of a are in meters and that of t is in seconds. Equations 27 and 30 and their time derivatives were substituted into Eq. 29, which has been transformed into a bubble trajectory equation as before. The new trajectory equation was solved numerically with the following initial conditions:

at $t = 0$ $U = dS/dt = 0.12$ m/s

at $t = 0$ $S = 0.15$ m

The results of the above analyses are shown in Figure 2, where the agreement is remarkable at the start of the bubble travel but a divergence is noticed towards the end of it. The main reason of this divergence was the departure of the growing bubble from the spherical shape, as was stated by Ohl et al. on page 729: "As revealed by the photos, the bubble remains essentially spherical until the middle of the expansion, but then it acquires a clear oblate shape." It should be noted that the time domain of the solution of Eq. 29 was restricted to the time from $t = 0$ to $t = 0.08$ s. The bubble radius remains almost constant at $t > 0.08$ s, as shown in their Figure 2.

As a suggestion for future work, it is desirable to verify experimentally the derived Eqs. 10, 12, and 15 of the virtual mass coefficients of the growing bubble at various *Ja* numbers.

The importance of the present results justifies the modest mathematical elaboration encountered during the derivation of C_V. The utilization of these results in evaluating the parameters of the thermal hydraulics consequences that result from the hypothetical loss of coolant accident in light water cooled and moderated nuclear reactors, would facilitate calculations by advanced computer codes like RELAP and RETRAN.

Conclusion

New equations have been derived in this article showing the effects of virtual mass forces on a growing or collapsing bubble. The present analytical results compared well with the available experimental results and theoretical solutions of various investigators.

It was found that the virtual mass coefficient of the time-dependent radius of the growing or collapsing bubble varies according to the following:

- It increases with the increase in Jakob number.
- It decreases with the increase in the dimensionless time.
- It decreases with the increase in the instantaneous *Pe* number.

Notation

a = bubble radius
C_D = drag coefficient
c = specific heat
C_V = virtual mass coefficient
F_B = buoyancy force
F_D = drag force
L = distance between centers of two bubbles
g = gravitational acceleration
h_{fg} = latent heat of evaporation
Ja = Jakob number ($\rho_f c \Delta T/\rho_g h_{fg}$)
KE = kinetic energy
m_b = mass of bubble
M = virtual mass of bubble
Pe = Pèclet number ($2aU/\kappa$)
Pr = Prandtl number (μc/k)
r = polar coordinate
Re = Reynolds number ($2a\rho_f U/\mu_f$)
t = time
t_o = bubble departure time
U = bubble velocity
V_r = velocity component in r coordinate

Greek letters

κ = thermal diffusivity
ΔT = temperature excess over boiling point ($T_f - T_{sat}$)
θ = angular coordinate
ρ = density
μ = absolute viscosity
τ = dimensionless time
ϕ = velocity potential

Subscripts

f = liquid
g = vapor

Literature Cited

1. Plesset MS, Zwick SA. The growth of vapor bubbles in super-heated liquids. *J Appl Phys.* 1954;25:493-500.
2. Forster MS, Zuber N. Growth of vapor bubble in a superheated liquid. *J Appl Phys*. 1954;25:474-478.
3. Scriven LE. On the dynamics of phase growth. *Chem Eng Sci.* 1959; 10:1-13.
4. Bankoff SG. Diffusion-controlled bubble growth. *Advances in Chem Eng*. 1966;6:1-60.
5. Mikic BB, Rohsenow WM, Griffith P. On bubble growth rates. *Int J Heat Mass Transfer*. 1970;13:657-662.
6. Wallis GB. Discussion of a paper. *Int J Multiphase Flow.* 1980;4:585-586.
7. Kendoush AA. The virtual mass of a spherical-cap bubble. *Phys Fluids.* 2003;15:2782-2785.
8. Askovic R. On an evaluation of the drag force of a growing vapor bubble at rectilinear accelerated ascension. *Theoretical and Applied Mechanics.* 2002;27:103-116.
9. Legendre D, Boree J, Magnaudet J. Thermal and dynamic evaluation of a spherical bubble moving steadily in a superheated or subcooled liquid. *Phys Fluids.* 1998;10:1256-1272.
10. Magnaudet J, Legendre D. The viscous drag force on a spherical bubble with a time-dependent radius. *Phys Fluids.* 1998;10:550-554.
11. Ohl CD, Tijink A, Prosperetti A. The added mass of an expanding bubble. *J Fluid Mech.* 2003;482:271-290.
12. Prosperetti A. Some considerations on the modeling of disperse multiphase flow by averaged equations. *JSME Int J.* 1999;12:573-585.
13. Kim L.-H. Modeling of bubble and drop formation in flowing liquids in terrestrial and microgravity environments. Ph.D. Thesis, Case Western Reserve University, Cleveland, OH 1992.
14. Divinis N, Karapantsios TD, Kostoglou M, Panoutsos CS, Bontozoglou V, Michels AC, Sneep MC, De Bruijn R, Lotz H. Bubbles growing in supersaturated solutions at reduced gravity. *AIChE J.* 2004;50:2369-2382.
15. Batchelor GK, Ed. *The Scientific Papers of Sir Geoffrey Ingram Taylor.* Cambridge: Cambridge University Press, Volume III, p. 337; 1963.
16. Batchelor GK. *An Introduction to Fluid Dynamics.* Cambridge: Cambridge Univ. Press; 1967:479.
17. Lamb H. *Hydrodynamics*, 6th ed. Cambridge: Cambridge Univ. Press; 1932.
18. Milne-Thomson LM. *Theoretical Hydrodynamics*, 5th ed. London: Macmillan; 1972.
19. Riznic TE, Kojasoy G, Zuber N. On the spherically phase change problem. *Int J Fluid Mech Research.* 1999;26:110-145.
20. Florschuetz LW, Chao BT. On the mechanics of vapor bubble collapse *ASME: J Heat Transfer.* 1965;87:209-220.
21. Stelson TE, Mavis FT. Virtual mass and acceleration in fluids. *ASCE Trans.* 1957;122:518-530.
22. Iverson HW, Balent R. A correlating modulus for fluid resistance in accelerated motion. *J Appl Phys.* 1951;22:324-328.
23. Kendoush AA. The drag force on a growing bubble. The 11th International Topical Meeting on Nuclear Reactor Thermal-Hydraulics (NURETH-11), Popes' Palace Conference Centre, Avignon, France, October 2-6, 2005.
24. Oguz HN, Prosperetti A. A generalization of the impulse and virial theorems with an application to bubble oscillations. *J Fluid Mech.* 1990;218:143-162.
25. Darby R. The dynamics of vapour bubbles in nucleate boiling. *Chem Eng Sci.* 1964;19:39-49.
26. Takahashi K, Endo H, Oyama KH, Mori S. A virtual mass and a drag coefficient for an oscillating particle. *J Chem Eng. Japan.* 1992;25: 683-686.
27. Kendoush AA. Hydrodynamic model for bubbles in a swarm. *Chem Eng Sci.* 2001;56:235-238.
28. Kang IS, Leal LG. The drag coefficient for a spherical bubble in a uniform streaming flow. *Phys Fluids.* 1988;31:233-237.
29. Moore DW. The rise of a gas bubble in a viscous liquid. *J Fluid Mech.* 1959;6:113-130.
30. Moore DW. The boundary layer on a spherical gas bubble. *J Fluid Mech.* 1963;16:161-172.

Appendix: Derivation of the Drag Force on the Growing Bubble

Assume that the fluid is incompressible, neglect the boundary layer separation at the bubble surface, assume that the bubble maintains a spherical shape during growth, and assume that the flow is laminar with spherical symmetry.

The growing vapor bubble is rising with an instantaneous velocity U assuming potential flow and the velocity potential on the growing bubble is given by Milne-Thomson[18] and Batchelor[16] as follows:

$$\phi = \Phi_1 + \Phi_2 \tag{A1}$$

where

$$\Phi_1 = \frac{1}{2} U a^3 \frac{\cos\theta}{r^2}$$

and

$$\Phi_2 = -\frac{a^2}{r}\frac{da}{dt}.$$

The radial component of the velocity is given by the following:

$$V_r = -\frac{\partial \phi}{\partial r} = U \cos \theta\left(\frac{a}{r}\right)^3 - \left(\frac{a}{r}\right)^2 \frac{da}{dt} \tag{A2}$$

The total rate of viscous dissipation is given by Taylor[15] and utilized by Kendoush[27] as follows:

$$E = \frac{1}{2}\int_0^{\pi} \mu_f\left[\frac{\partial}{\partial r}(V_r)^2\right]_{r=a} 2\pi a^2 \sin \theta d\theta \tag{A3}$$

Substituting Eq. A2 into Eq. A3 and completing the integration leads to the following:

$$E = -4\pi a\mu_f\left[U^2 + 2\left(\frac{da}{dt}\right)^2\right] \tag{A4}$$

The drag force F_D over the surface of the growing bubble is given as:

$$F_D = E/U = -4\pi a\mu_f\left[U + \frac{2}{U}\left(\frac{da}{dt}\right)^2\right] \tag{A5}$$

The negative sign is normally neglected in this equation. The first term on the right side of this equation represents the drag force due to the viscous effects of the rising bubble, and the second term represents the drag force due to bubble growth that arises from the momentum associated with the radial flow of liquid surrounding the growing bubble. This is a new equation that has not been reported before. The drag coefficient of the growing bubble may be defined as follows:

$$C_D = \frac{F_D}{0.5\rho_f U^2 \pi a^2} \tag{A6}$$

Substituting Eq. A5 into Eq. A6 gives:

$$C_D = \frac{16}{\mathrm{Re}}\left(1 + \frac{2}{U^2}\left[\frac{da}{dt}\right]^2\right) \tag{A7}$$

As the growth component of this equation, that is, $da/dt \rightarrow 0$, this equation reduces to that given by Kang and Leal[28] (that is, $C_D = 16/\mathrm{Re}$). This drag is entirely due to pressure forces on a fixed radius bubble. This drag, together with the viscous drag derived by Moore[29] (that is, $C_D = 32/\mathrm{Re}$), constitutes the total drag of Moore[30] (that is, $C_D = 48/\mathrm{Re}$). This distribution of the drag on the surfaces of a fixed radius bubble was also appreciated by Legendre et al.[9] It should be noted that the $C_D = 16/\mathrm{Re}$ is originally credited to Hadamard and Rybczynski as an equation for the drag coefficient of a single bubble at $\mathrm{Re} << 1.0$.

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